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Boolean rank of Kronecker products

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Abstract

The Boolean rank of an $m \times n$ binary matrix A is the least integer k such that A is the product of $m \times k$ and $k \times n$ binary matrices, under Boolean arithmetic. The product of the Boolean ranks of two matrices A and B is an upper bound on the Boolean rank of their Kronecker product. An example is given to show that this bound need not be tight. © 2001 Elsevier Science Inc. All rights reserved.

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Throughout, all matrices are Boolean. That is, each matrix is binary and arithmetic is as usual except $1 + 1 = 1$. For background information on Boolean matrices see [5]. The *Boolean rank*, $r_{\mathrm{B}}(A)$, of an $m \times n$ matrix A is the least integer k such that $A = BC$, where B is $m \times k$ and C is $k \times n$. Boolean rank is also known as *Schein rank* [5]. By convention, the Boolean rank of the all-zeroes matrix is zero. Alternatively, $r_{\mathrm{B}}(A)$ may be defined as the minimum number of Boolean rank 1 matrices uv^{T} that sum to A under Boolean arithmetic; that is, $r_{\mathrm{B}}(A)$ is the minimum number of all-ones submatrices of A that cover all of the ones of A. It follows from the alternate definition that for all $m \times n$ matrices A:

1. $r_{\mathrm{B}}(A) \leqslant \min\{m, n\}$;
2. $r_{\mathrm{B}}(A) = r_{\mathrm{B}}(A^{\mathrm{T}})$;
3. $r_{\mathrm{B}}(AB) \leqslant \min\{r_{\mathrm{B}}(A), r_{\mathrm{B}}(B)\}$ for all $n \times k$ matrices B;
4. $r_{\mathrm{B}}(B) \leqslant r_{\mathrm{B}}(A)$ for all submatrices B of A.

For other results regarding Boolean rank, see [1,2,4,5] and for more recent surveys, see [3,6].

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PII: S 0 0 2 4 - 3 7 9 5 (0 1) 0 0 3 3 8 - X

In [7], Orlin provided a graph-theoretic interpretation of Boolean rank. For a bipartite graph G with bipartition $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$ the *bipartite adjacency matrix* of G is the $m \times n$ binary matrix whose ijth entry is 1 if x_i is adjacent to y_j and 0 otherwise. The Boolean rank of an $m \times n$ matrix A is the minimum number of complete bipartite subgraphs covering all of the edges of the bipartite graph G whose bipartite adjacency matrix is A. Orlin also showed that the problem of determining $r_{\mathrm{B}}(A)$ is NP-complete. For more on this graphical interpretation, see [6,7].

In an $m \times n$ matrix A, row i is dominated by row j if $A_{ik} \leqslant A_{jk}$ for all $k = 1, \ldots, n$. The matrix A has *row-domination* if and only if, for some $i \neq j$, row i is dominated by row j. That is, A has row-domination means that for some $i \neq j$, $A_{ik} = 1$ implies $A_{jk} = 1$ for all $k = 1, \ldots, n$. For an $m \times n$ matrix A, define the *complement* of A to be the $m \times n$ matrix $\bar{A}$ obtained by interchanging the zeroes and ones in A. In particular, $\bar{I}_n$ denotes the $n \times n$ matrix with zeroes on the main diagonal and ones everywhere else. Theorem 1 appears as Corollaries 1 and 2 in [1], where the proof employs a lemma of Sperner.

Theorem 1. *Let A be an $m \times n$ binary matrix. If A does not have row-domination, then $r_{\mathrm{B}}(A) \geqslant s(m)$, where*

$$s(m) = \min\left\{k : m \leqslant \binom{k}{\lfloor \frac{k}{2} \rfloor}\right\}.$$

If A^{T} does not have row-domination, then $r_{\mathrm{B}}(A) \geqslant s(n)$. Furthermore, $r_{\mathrm{B}}(\bar{I}_n) = s(n)$.

As in [4], a set of ones of A is *isolated* if no pair of ones are in an all-ones submatrix of A together. Let $i(A)$ be the maximum number of ones in an isolated set of A. The alternate definition of $r_{\mathrm{B}}(A)$ leads immediately to the bound $r_{\mathrm{B}}(A) \geqslant i(A)$. The *Kronecker product* of an $m \times n$ matrix A and a $p \times q$ matrix B is the $mp \times nq$ matrix $A \otimes B$ which can be expressed as an $m \times n$ block matrix with the ijth block being B if $A_{ij} = 1$ and a zero block otherwise. Theorem 2 appears in [2] and provides bounds on the Boolean rank of the Kronecker product of two matrices.

Theorem 2. *Let A and B be Boolean matrices. Then*
1. $\max\{i(A)r_{\mathrm{B}}(B), r_{\mathrm{B}}(A)i(B)\} \leqslant r_{\mathrm{B}}(A \otimes B) \leqslant r_{\mathrm{B}}(A)r_{\mathrm{B}}(B)$;
2. $i(A)i(B) \leqslant i(A \otimes B) \leqslant \min\{i(A)r_{\mathrm{B}}(B), r_{\mathrm{B}}(A)i(B)\}$.

The authors of [2] did not find an example where $r_{\mathrm{B}}(A \otimes B) < r_{\mathrm{B}}(A)r_{\mathrm{B}}(B)$, although they suggested $\bar{I}_n \otimes \bar{I}_n$ as a possible candidate. Note that $i(\bar{I}_4) = 3$ and $r_{\mathrm{B}}(\bar{I}_4) = 4$, so Theorem 2 implies $12 \leqslant r_{\mathrm{B}}(\bar{I}_4 \otimes \bar{I}_4) \leqslant 16$. Using Theorem 3 below, it is possible to show that, in fact, $r_{\mathrm{B}}(\bar{I}_4 \otimes \bar{I}_4) = 12$. A careful justification shows that $r_{\mathrm{B}}(A) = i(A)$ for all $m \times n$ matrices A with $1 \leqslant m, n \leqslant 4$ and at most one of m and n is 4. Consequently, $\bar{I}_4 \otimes \bar{I}_4$ is the smallest such example in terms of order.

Before stating Theorem 3 and the construction which gives $r_{\rm B}(\bar{I}_4 \otimes \bar{I}_4) = 12$, some new terminology is necessary. For a Boolean rank 1 matrix $A = uv^{\rm T}$, define the *opposite* of A to be the Boolean rank 1 matrix $\tilde{A} = \bar{u}\bar{v}^{\rm T}$.

Theorem 3. *Let A be an $m \times n$ binary matrix. Suppose there exists a set $\mathscr{M}$ of Boolean rank-*1 *matrices with the properties*:
1. $\sum_{M\in\mathscr{M}} M = A$ *under Boolean arithmetic*;
2. $M \in \mathscr{M}$ *implies* $\tilde{M} \in \mathscr{M}$;
3. *For each* (i, j) *with* $A_{ij} = 1$,

$$\sum_{M\in\mathscr{M}, M_{ij}=1} (M + \tilde{M}) = A$$

under Boolean arithmetic.

Then $r_{\rm B}(A \otimes A) \leqslant 2|\mathscr{M}|$.

Proof. By Theorem 2, $r_{\rm B}(M \otimes M) = r_{\rm B}(M \otimes \tilde{M}) = 1$ for each $M \in \mathscr{M}$. Thus, to show $r_{\rm B}(A \otimes A) \leqslant 2|\mathscr{M}|$, it suffices to show that $A \otimes A$ and $\sum_{M\in\mathscr{M}}[(M \otimes M) + (M \otimes \tilde{M})] = \sum_{M\in\mathscr{M}} M \otimes (M + \tilde{M})$ are the same matrix. This can be accomplished by showing these two matrices agree block by block.

The *ij*th block of $A \otimes A$ is $A_{ij}A$ and is either a zero block or A. The *ij*th block of $\sum_{M\in\mathscr{M}} M \otimes (M + \tilde{M})$ is $\sum_{M\in\mathscr{M}} M_{ij}(M + \tilde{M})$ and so by property (3) is either a zero block or A. Since $A_{ij} = 0$ if and only if $M_{ij} = 0$ for all $M \in \mathscr{M}$, it follows that the *ij*th block of $A \otimes A$ is a zero block if and only if the *ij*th block of $\sum_{M\in\mathscr{M}} M \otimes (M + \tilde{M})$ is a zero block. Similarly, $A_{ij} = 1$ if and only if $M_{ij} = 1$ for some $M \in \mathscr{M}$ and consequently the *ij*th block of $A \otimes A$ is non-zero (and hence A) if and only if the *ij*th block of $\sum_{M\in\mathscr{M}} M \otimes (M + \tilde{M})$ is non-zero (and hence A). □

To use Theorem 3 on $\bar{I}_4$, consider the following six Boolean rank-1 matrices:

$$\mathscr{M} = \left\{ \begin{bmatrix} 0&0&1&1\\ 0&0&1&1\\ 0&0&0&0\\ 0&0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0&0\\ 0&0&0&0\\ 1&1&0&0\\ 1&1&0&0 \end{bmatrix}, \begin{bmatrix} 0&1&0&1\\ 0&0&0&0\\ 0&1&0&1\\ 0&0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0&0\\ 1&0&1&0\\ 0&0&0&0\\ 1&0&1&0 \end{bmatrix}, \begin{bmatrix} 0&1&1&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&1&1&0 \end{bmatrix}, \begin{bmatrix} 0&0&0&0\\ 1&0&0&1\\ 1&0&0&1\\ 0&0&0&0 \end{bmatrix} \right\}.$$

This set $\mathscr{M}$ satisfies the three conditions of Theorem 3 for $\bar{I}_4$. Consequently, $r_{\rm B}(\bar{I}_4 \otimes \bar{I}_4) = 12$.

Since the matrices in $\mathscr{M}$ of Theorem 3 occur in pairs, $|\mathscr{M}|$ is even. Also, because every pair of ones in an isolated set of A must be in a distinct matrix/opposite pair, it follows that $|\mathscr{M}| \geqslant i(A)(i(A) - 1)$. Note that for $\bar{I}_4$ this bound is attained.

Although Theorem 3 provides an upper bound on $r_{\mathrm{B}}(A \otimes A)$, this bound will only be an improvement on the bound given in Theorem 2 when $|\mathscr{M}| \leqslant \frac{1}{2} r_{\mathrm{B}}(A)^2$.

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